Fuzzy Riemann Surfaces

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Abstract. *We introduce C-Algebras (quantum analogues of compact Riemann surfaces), defined by polynomial relations in non-commutative variables and containing a real parameter that, when taken to zero, provides a classical non-linear, Poisson-bracket, obtainable from a single polynomial C(onstraint) function. For a continuous class of quartic constraints, we explicitly work out finite dimensional representations of the corresponding C-Algebras.*

Introduction

Harmonic homogenous polynomials in 3 commuting variables, upon substitution of N-dimensional representations of $\mathfrak{su}(2)$ for the commuting variables, can be used to define a map from functions on $\mathbb{S}^2$ to $N \times N$ matrices, that sends Poisson brackets to matrix commutators [GH82]. The result was dubbed "Fuzzy Sphere", in [Mad92]. In [KL92] and [BMS94] it was proven (conjectured in [BHSS91]) that the (complexified) Poisson algebra of functions on *any* Riemann surface arises as a $N \to \infty$ limit of $\mathfrak{gl}(N, \mathbb{C})$. Insight on how matrices can encode topological information (certain sequences having been indentifyable as converging to a particular function, but $\mathfrak{gl}(N, \mathbb{C})$ lacking topological invariants) was gained in [Shi04]. That no concrete anologue of the Fuzzy Sphere construction [GH82] for higher genus compact surfaces could be found, and that the one found for the torus [FFZ89, Hop88] was of a very different nature, remained an unsolved puzzle, just as the rigidity of the 2 constructions. We are happy to announce a resolution by presenting a unified (*and* concrete, as well as non-rigid, and intuitively simple) treatment for general compact Riemann surfaces. In section 1 we describe Riemann surfaces of genus g embedded in $\mathbb{R}^3$ as inverse images of polynomial contraint-functions, $C(\vec{x})$. In section 2 we define a Poisson bracket on $\mathbb{R}^3$, to be restricted to the embedded Riemann surface. In section 3 we review the quantization for the round 2-sphere. In section 4 we outline our construction for general genus. In section 5 we work out the conjectured construction for a continuous class of tori and deformed spheres. In section 6 we discuss how the classical singularity at $\mu = 1$ is reflected in the quantum world.

1 Genus g Riemann surfaces

The aim of this section is to present compact connected Riemann surfaces of any genus embedded in $\mathbb{R}^3$ by inverse images of polynomials. For this purpose we use the regular value theorem and Morse theory. Let C be a polynomial in 3 variables and define $\Sigma = C^{-1}(\{0\})$. What are the conditions on C, for Σ to be a genus g Riemann surface? If C is a submersion on Σ, then Σ is an orientable submanifold of $\mathbb{R}^3$. Σ has to be compact and of the desired genus. For further details see [Hir76, Hof02].

The classification of 2 dimensional compact (connected) manifolds is well known. In this case, there is a one to one correspondance between topological and diffeomorphism classes. The result is that any compact orientable surfaces is homeomorphic (hence diffeomorphic) to a sphere or to a surface obtained by glueing tori together (connected sum). The number g of tori is called the genus and is related to the Euler-Poincaré characteristic by the formula $\chi = 2 - 2g$.

Morse theory To compute $\chi(\Sigma)$ we apply Morse theory to a specific function. A point p of a (smooth) function f on Σ is a singular point if $Df_p = 0$, in which case $f(p)$ is a singular value. At any singular point p one can consider the second derivative $D^2 f_p$ of f and p is said to be non-degenerate if $\det(D^2 f_p) \neq 0$. Moreover one can attach an index to each such point depending on the signature of $D^2 f$: 0 if positive, 1 if hyperbolic and 2 if negative. A Morse function is a function such that every singular point is non-degenerated and singular values all distinct. Then $\chi(\Sigma)$ is given by the formula:

$$\chi(\Sigma) = n(0) - n(1) + n(2),$$

where $n(i)$ is the number of singular points which have an index i.

The Cote_x function is defined as the restriction of the first projection on the surface. It's not necessarily a Morse function (one has to choose a "good" embedding for that), but the singular points are those for which the gradient $\operatorname{grad} C$ is parallel to the Ox axis. Moreover the Hessian matrix of Cote_x at such a point p is:

$$-\frac{1}{\frac{\partial C}{\partial x}(p)} \begin{pmatrix} \frac{\partial^2 C}{\partial y^2}(p) & \frac{\partial^2 C}{\partial y \partial z}(p) \\ \frac{\partial^2 C}{\partial y \partial z}(p) & \frac{\partial^2 C}{\partial z^2}(p) \end{pmatrix}.$$

Polynomial model Take

$$C(\vec{x}) = (P(x) + y^2)^2 + z^2 - \mu^2,$$

where $\mu > 0$, $P(x) = a_{2k}x^{2k} + a_{2k-1}x^{2k-1} + \cdots + a_1 x + a_0$ with $a_{2k} > 0$ and $k > 0$. Obviously Σ is closed and bounded (even degree of P) hence compact. Σ is a submanifold of $\mathbb{R}^3$ if, and only if for each $p \in \Sigma$, $DC_p \neq 0$ which is equivalent to requiring that the polynomials $P - \mu$ and $P + \mu$ have only simple roots. The singular points of the Cote_x function on Σ are the points $(x, 0, 0)$ such that $P(x)^2 = \mu^2$ and the Hessian matrix is:

$$-\frac{1}{\frac{\partial C}{\partial x}(x,0,0)} \begin{pmatrix} 4P(x) & 0 \\ 0 & 2 \end{pmatrix}.$$

Hence it is positive or negative if, and only if $P(x) = \mu$ and hyperbolic if, and only if $P(x) = -\mu$. With the fact that $P(x)$ can't be zero at a singular point, it also proofs that Cote_x is a genuine Morse function. Finally,

$$n(0) + n(2) = \#\{P = \mu\} \qquad \text{and} \qquad n(1) = \#\{P = -\mu\}.$$

If the polynomial $P - \mu$ has exactly 2 simple roots and the polynomial $P + \mu$ has exactly $2g$ simple roots, then $\chi(\Sigma) = 2 - 2g$ and Σ is a surface of genus g.

Explicit construction of P Let $g > 0$. Set:

(i) $\quad G(t) = (t-1)(t-2^2)\dots(t-g^2) \quad$ and $\quad M = \max_{0 \le t \le g^2+1} G(t), \quad \alpha \in \left(0, \frac{2\mu}{M}\right)$

(ii) $\quad Q(x) = \alpha G(x) - \mu \quad$ and $\quad P(x) = Q(x^2)$

One can directly see that $Q + \mu$ has exactly g simple roots, hence $P + \mu$ has exactly $2g$ simple roots. For $t \in [0; g^2 + 1]$, the function $Q(t) - \mu$ has no zero. On the other hand, for $t \geq g^2 + 1$, $Q(t) - \mu$ is strictly growing and has exacly one zero. Consequently the polynomial $P - \mu$ has exactly 2 simple roots and the surface Σ defined above is a genus g compact Riemann surface.

2 Poisson brackets in $\mathbb{R}^3$

For arbitrary $C : \mathbb{R}^3 \longrightarrow \mathbb{R}$ (twice continuously differentiable)

$$\{f, g\}_{\mathbb{R}^3} := \vec{\nabla} C \cdot (\vec{\nabla} f \times \vec{\nabla} g) \tag{2.1}$$

defines a Poisson bracket for functions on $\mathbb{R}^3$ (see e.g. Nowak [Now97] who studied the formal deformability of (2.1))[1]. Let $\Sigma_g \subset \mathbb{R}^3$ be described, as in section 1, by a "constraint":

$$\frac{1}{2} C(\vec{x}) := \psi(x, y) + \frac{z^2 - 1}{2} \stackrel{!}{=} 0. \tag{2.2}$$

The Poisson brackets between x,y and z then read:

$$\begin{aligned} \{x, y\}_{\mathbb{R}^3} &= z \\ \{y, z\}_{\mathbb{R}^3} &= \psi_x \\ \{z, x\}_{\mathbb{R}^3} &= \psi_y. \end{aligned} \tag{2.3}$$

Explicitely, substituting $\{x, y\}$ for z, one obtains

$$\psi(x, y) + \frac{1}{2}\{x, y\}^2_{\mathbb{R}^3} = \text{const} \left(= \frac{1}{2}\right), \tag{2.4}$$

resp.

$$\begin{aligned} \psi_x &= \{y, \{x, y\}_{\mathbb{R}^3}\}_{\mathbb{R}^3} \\ \psi_y &= \{\{x, y\}_{\mathbb{R}^3}, x\}_{\mathbb{R}^3}. \end{aligned} \tag{2.5}$$

Let $x(\sigma_1, \sigma_2)$, $y(\sigma_1, \sigma_2)$, $z(\sigma_1, \sigma_2)$ be a local parametrisation of Σ_g. Restricting $\{f, g\}_{\mathbb{R}^3}$ to a Poisson bracket, $\{f, g\}_C$, on the surface $C(\vec{x}) = 0$, and realizing $\{f, g\}_C$ on Σ_g locally as

$$\frac{1}{\rho(\sigma^1, \sigma^2)} \left(\frac{\partial f}{\partial \sigma^1} \frac{\partial g}{\partial \sigma^2} - \frac{\partial g}{\partial \sigma^1} \frac{\partial f}{\partial \sigma^2} \right),$$

the relation (equivalence!, up to different constant values on the r.h.s. of (2.4)) between (2.4) and (2.5) is seen as follows: differentiating (2.4) with respect to the local parameters, φ^1 and φ^2, one obtains a linear system of equations for ψ_x and ψ_y, whose algebraic solution (via Cramers rule, e.g.) gives (2.5). To go from (2.5) to (2.4) (with the constant unspecified, of course) one either notes simply that the l.h.s. of (2.4) commutes with both x and y (according to (2.5)), or one directly solves (2.5) via a hodograph-transformation (cp. [BKL05], in which Poisson bracket equations are considered, whose solutions also contain surfaces of general type[2]); changing independent variables from σ^1, σ^2 to $x_1 = x(\sigma^1, \sigma^2)$ and $x_2 = y(\sigma^1, \sigma^2)$; using (deriving) (as e.g. in [BH93])

$$\begin{aligned} \{x, y\}_C &=: J(x_1, x_2) \\ \{f, x\}_C &= -J f_y \\ \{f, y\}_C &= J f_x, \end{aligned}$$

(2.5) then becomes

$$\begin{aligned} -J J_x &= \psi_x \\ -J J_y &= \psi_y, \end{aligned}$$

i.e. $\frac{1}{2} J^2 + \psi = \text{const}$.

[1] While we did not (yet find a way to) use his results, we are very grateful for his "New Year's Eve" explanations, as well as providing us with his Ph.D. Thesis.

[2] Apart from spheres, however, these surfaces are either non-polynomial, or non-compact – or both – causing the corresponding quantum-algebras to be necessarily *different* from ours.

3 The fuzzy sphere [GH82]

Consider the usual spherical harmonics,

$$\left\{Y_{lm}(\theta,\varphi)\right\}_{\substack{l=1,\ldots,\infty\\ m=-l,\ldots,+l}},$$

eigenfunctions of the Laplace operator on $\mathbb{S}^2$ ($\triangle_{\mathbb{S}^2} Y_{lm} = -l(l+1)Y_{lm}$). Write them as harmonic homogenous polynomials in $x_1 = r\sin\theta\cos\varphi$, $x_2 = r\sin\theta\sin\varphi$ and $x_3 = r\cos\theta$ (restricted to $r^2 = \vec{x}^2 = 1$):

$$Y_{lm}(\theta,\varphi) = \sum c^{(m)}_{a_1 a_2\cdots a_l}\; x_{a_1} x_{a_2}\cdots x_{a_l} \tag{3.1}$$

(where the tensor $c_{\ldots}$ is by definition traceless and totally symmetric), and then replace the commuting variables x_a by generators X_a of the N-dimensional irreducible (spin $s = \frac{N-1}{2}$) represensation of $\mathfrak{su}(2)$, to obtain N^2-1 $N\times N$-matrices:

$$T^{(N)}_{lm} := \gamma_{Nl}\sum c^{(m)}_{a_1 a_2\cdots a_l}\; X_{a_1}X_{a_2}\cdots X_{a_l} \qquad \text{for} \quad l = 1,\ldots,N-1 \quad m=-l,\ldots,+l\,; \tag{3.2}$$

automatically, $T^{(N)}_{lm} \equiv 0$ for $l \geq N$. Instead of having $\vec{X}^{\,2} := X_1^2 + X_2^2 + X_3^2$ equal to $\frac{N^2-1}{4}\mathbb{1}$ (the usual normalisation), it is advantageous to choose the normalisation $\vec{X}^2 = \mathbb{1}$,

$$[X_a, X_b] = \frac{2i}{\sqrt{N^2-1}}\epsilon_{abc}X_c, \tag{3.3}$$

and then $\gamma_{Nl} = -i\sqrt{\frac{N^2-1}{4}}$. As the Poisson bracket on $\mathbb{S}^2$,

$$\{f,g\}_{\mathbb{S}^2}(\theta,\varphi) := \frac{1}{\sin\theta}\left(\frac{\partial f}{\partial\theta}\frac{\partial g}{\partial\varphi} - \frac{\partial g}{\partial\theta}\frac{\partial f}{\partial\varphi}\right) \tag{3.4}$$

can be obtained by restricting the Poisson bracket

$$\{f,g\}_{\mathbb{R}^3}(\vec{x}) := \vec{x}\cdot\left(\vec{\nabla}f(\vec{x})\times\vec{\nabla}g(\vec{x})\right) \tag{3.5}$$

to $\mathbb{S}^2$ (via $\vec{x}^{\,2} = 1$), $\{Y_{lm}, Y_{l'm'}\}_{\mathbb{S}^2}$ can be computed from

$$\left\{r^l Y_{lm}, r^{l'}Y_{l'm'}\right\}_{\mathbb{R}^3} = \sum c^{(m)}_{a_1\ldots a_l}\; c^{(m')}_{b_1\ldots b_{l'}}\;\left\{x_{a_1}x_{a_2}\cdots x_{a_l}, x_{b_1}x_{b_2}\cdots x_{b_{l'}}\right\} \tag{3.6}$$

by using the derivation property, and

$$\{x_a, x_b\} = \epsilon_{abc}x_c \tag{3.7}$$

(following from (3.5)), as well as then decomposing the resulting polynomial of degree $l+l'-1$ into harmonic homogenous ones, to obtain the structure constants of the Lie-Poisson algebra of functions on the 2-sphere (in the basis of the spherical harmonics).
Calculating

$$\left[T^{(N)}_{lm}, T^{(N)}_{l'm'}\right] = -\frac{N^2-1}{4}\sum c^{(m)}_{a_1\ldots a_l}\; c^{(m')}_{b_1\ldots b_{l'}}\;\left[X_{a_1}X_{a_2}\cdots X_{a_l}, X_{b_1}X_{b_2}\cdots X_{b_{l'}}\right] \tag{3.8}$$

the first step is identical to the one after (3.6), while any further use of the commutation relations (3.3) – necessary to obtain the desired traceless totally symmetric tensors – induces factors of $1/\sqrt{N^2-1}$; hence one finds agreement to leading order of N of the structure constants of $\mathfrak{gl}(N,\mathbb{C})$, in the basis $\left\{T^{(N)}_{lm}\right\}_{l=1,\ldots,N-1}$, satisfying

$$\mathrm{Tr}\left(T^{(N)\dagger}_{lm}T^{(N)}_{l'm'}\right) = \delta_{ll'}\delta_{mm'}\frac{(N+l)!}{16\pi(N-1-l)!(N^2-1)^{l-1}},$$

with those of the Poisson algebra.

4 The construction for general Riemann surfaces

Let us consider compact Riemann surfaces $\Sigma_g \in \mathbb{R}^3$ described by

$$\left(P(x)+y^2\right)^2+z^2=\mathrm{const}\,(=1) \tag{4.1}$$

(with P, as in section (1), an even polynomial of degree $2g$), resp.

$$\begin{aligned}\{y,\{x,y\}\} &= P'(x)(P(x)+y^2)\\ \{\{x,y\},x\} &= 2y(P(x)+y^2)\end{aligned} \tag{4.2}$$

(for (2.3)). We claim that fuzzy analogues of Σ_g can be obtained via matrix analogues of (4.1) and (4.2). Apart from possible “explicit $1/N$ corrections”, direct ordering questions arise both on the r.h.s. of (4.2), and in (4.1), while on the l.h.s. of (4.2) one replaces Poisson brackets by $\frac{1}{i\hbar}$(commutator(s)). Consider therefore the problem of looking for matrices X, Y satisfying

$$(P(X)+Y^2)^2-\frac{1}{\hbar^2}[X,Y]^2=\mathbb{1}, \tag{4.3}$$

resp.

$$\frac{1}{\hbar^2}[X,[X,Y]] = 2Y^3+YP(X)+P(X)Y \tag{4.4}$$

$$\frac{1}{\hbar^2}[Y,[Y,X]] = \sum_{r=1}^{2g-1} a_r\sum_{i=0}^{r-1}X^i\left(P(X)+Y^2\right)X^{r-1-i} \tag{4.5}$$

if $P(X)=\sum_{r=0}^{2g}a_rX^r$; the r.h.s. of (4.5) will also be denoted by $P(X)'|_{\varphi=P(X)+Y^2}$ (for a term X^4 in $P(X)$, e.g. , $P'(X)|_\varphi$ would correspondingly include $X^3\varphi+X^2\varphi X+X\varphi X^2+\varphi X^3$). This ordering in (4.4) and (4.5) is consistent, as

$$\begin{aligned}\big([X,Y]Y\big)X-[X,Y](YX) &= \Big(Y[X,Y]+\big[[X,Y],Y\big]\Big)X-[X,Y]\big(XY+[Y,X]\big) && (4.6)\\ &= \cdots\\ &= \Big[Y,\big[[X,Y],X\big]\Big]+\Big[\big[[X,Y],Y\big],X\Big] && (4.7)\end{aligned}$$

indeed equals to zero (insert (4.4) and (4.5) for the 2 double commutators to get $P(X)Y^2-Y^2P(X)+[P'(X)|_{Y^2},X]=\cdots=0$), which is has to, due to associativity of matrix multiplication (and resulting Jacobi identity).

Finding (for specific values of $\hbar^2$) concrete representations of (4.4) and (4.5), resp.(4.3), let alone classifying them, is of course a very complicated task. We succeeded in doing so for $P(x)=x^2-\mu$ (corresponding to a torus when $\mu>1$, and deformed spheres, when $-1<\mu<1$, see section (5), but first we would like to outline some qualitative features involved for the general case. As mentioned in the introduction, one of the puzzles was the rigidity of the construction for the round 2-sphere [GH82] and the toral rational rotation algebra [FFZ89] (see also [Hop89]). In the present construction we now have a (generally, i.e. apart from certain critical values – signaling topology change) continuous dependence on the data (P) which describe the Riemann surface. For given P, and $\hbar$, we define the corresponding Fuzzy Riemann Surface $\Sigma_\hbar(P)$ (for fixed P, expected to exist for infinitely many discrete values of $\hbar$, coming arbitrarily close to zero) as the algebra generated by the corresponding finite (N-)dimensional solutions (X,Y) of (4.4),(4.5), resp.(4.3). According to the semiclassical philosophy explained below (cp. [Shi04]) X and Y will exhibit eigenvalue sequencies (generically smoothly depending on P) characteristic of the topological type, reflecting the behaviour of the corresponding classical embedding functions x and y.

Let us observe that we can read off information of topology from a generic single function f by using Morse theory. This Morse theoretic information of topology manifests itself in the eigenvalue distribution of the matrix $\hat{f}$ corresponding to the function f:

the key idea is to introduce an auxiliary Hamiltonian dynamical system, whose phase space is the surface and whose Hamiltonian is given by f. Thus we consider ordinary differential equations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\sigma^1 &= \{\sigma^1, f\}, \\ \frac{\mathrm{d}}{\mathrm{dt}}\sigma^2 &= \{\sigma^2, f\}, \end{aligned} \tag{4.8}$$

where σ^1, σ^2 parametrise the embedded surface. Since classical orbits of (4.8) are equal-f lines on the surface, the family of the classical orbits exhibits branching processes which exactly reflect Morse theoretic information. The eigenvalue distribution of $\hat{f}$ is determined (to leading order in $\frac{1}{N}$) by the Bohr-Sommerfeld rule. That is, the eigenvalues are values of f on classical orbits which are such that the area between adjacent orbits is equal to the total area of the surface divided by N. It follows that the eigenvalues are grouped into subsets each corresponding to the branches on the surface. These subsets are called in [Shi04] as eigenvalue sequences, and have the property that (for sufficiently large N) the eigenvalues belonging to a sequence rise in a smooth way. The branching processes of the sequences are the same as those of the classical orbits. Furthermore, using the eigenvalue sequences and (4.8), a rule to calculate general (off-diagonal) matrix elements of a matrix $\hat{g}$ corresponding to a general function g was given in [Shi04]. Essential properties of the matrix regularisation, such as the correspondence of the matrix commutator and the Poisson bracket, can be derived from those rules.

5 Explicit solutions for tori and deformed spheres: Representations of the simplest non-linear C–Algebras

Let now $P(x) = x^2 - \mu$ which, for $\mu > 1$ describes a torus, and for $-1 < \mu < 1$ a (deformed) sphere. We will construct solutions for the corresponding matrix equations, which we take as

$$[X, Y] = i\hbar Z \tag{5.1}$$

$$[Y, Z] = i\hbar\{X, X^2 + Y^2 - \mu\} \tag{5.2}$$

$$[Z, X] = i\hbar\{Y, X^2 + Y^2 - \mu\} \tag{5.3}$$

$$\left(X^2 + Y^2 - \mu\right)^2 + Z^2 = \mathbb{1} \tag{5.4}$$

(in this section, $\{A, B\}$ denotes the anti-commutator $AB+BA$, and *not* a Poisson-bracket). Denoting $X^2 + Y^2 - \mu$ by φ, one finds (using the Jacobi-identity, the derivation property, and equations (5.2) + (5.3)) that

$$\begin{aligned} [\varphi, Z] &= \{X, [X, Z]\} + \{Y, [Y, Z]\} \\ &= -i\hbar\{X, \{Y, \varphi\}\} + i\hbar\{Y, \{X, \varphi\}\} \\ &= i\hbar\Big[\varphi, [X, Y]\Big] = -\hbar^2[\varphi, Z], \end{aligned} \tag{5.5}$$

hence $\varphi(X, Y)$ and Z commute, and can be diagonolized simultaneously; it then also follows that $\varphi^2 + Z^2$ is central, i.e. commutes with X, Y and Z.

In complex notation, $W := X + iY$, (5.2) and (5.3) can be written together as

$$\left(W^2W^\dagger + W^\dagger W^2\right)(\hbar^2 + 1) = 4\mu\hbar^2 W + 2(1 - \hbar^2)WW^\dagger W, \tag{5.6}$$

from which the crucial commutativity of $D := WW^\dagger$ and $\widetilde{D} := W^\dagger W$,

$$\begin{aligned} \left[WW^\dagger, W^\dagger W\right] &= \left[X^2 + Y^2 - i[X, Y], X^2 + Y^2 + i[X, Y]\right] \\ &= 2i\Big[X^2 + Y^2, [X, Y]\Big] = 0, \end{aligned} \tag{5.7}$$

also follows directly (by using (5.6)). In the basis where both D and $\widetilde{D}$ are diagonal

$$\begin{aligned} D &= \mathrm{diag}(d_1, \dots, d_N) \\ \widetilde{D} &= \mathrm{diag}(\widetilde{d}_1, \dots, \widetilde{d}_N), \end{aligned}$$

(5.6) becomes

$$W_{ij}\Big((\hbar^2+1)(\tilde d_i+d_j)+(\hbar^2-1)(d_i+\tilde d_j)-4\mu\hbar^2\Big)=0, \tag{5.8}$$

and from $(WW^\dagger)W=W(W^\dagger W)$ we obtain

$$W_{ij}(d_i-\tilde d_j)=0 \tag{5.9}$$

(which in fact was already used when writing (5.8) "symmetrically"). Thus, for $W_{ij}\neq 0$, (5.8) and (5.9) are equivalent to

$$\begin{gathered}\begin{pmatrix} d_j\\ \tilde d_j\end{pmatrix}=\begin{pmatrix}\alpha & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix} d_i\\ \tilde d_i\end{pmatrix}+\begin{pmatrix}\delta\\ 0\end{pmatrix}\\ \alpha=2\frac{1-\hbar^2}{1+\hbar^2}=2\cos 2\theta \quad , \quad \delta=4\mu\frac{\hbar^2}{1+\hbar^2}=4\mu\sin^2\theta,\end{gathered} \tag{5.10}$$

where we have put $\hbar=\tan\theta$. (5.10) is of the form

$$\vec x_j=A\vec x_i+\vec c \quad ; \tag{5.11}$$

so, for

$$W=\begin{pmatrix} 0 & w_1 & 0 & \cdots & 0\\ 0 & 0 & w_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & w_{N-1}\\ w_N & 0 & \cdots & 0 & 0\end{pmatrix}, \quad w_k\neq 0, \tag{5.12}$$

$$\vec x^{(n+1)}=A\vec x^{(n)}+\vec c, \quad n=1,2,\ldots,N \tag{5.13}$$

are the only equations relating the N eigenvalue pairs. What can we say about the possibility of coming back to the same point $\vec x$ after N iterations, i.e

$$\vec x^{(N+1)}-\vec x=A^N\vec x+\Big(\mathbb{1}+A+A^2+\cdots+A^{N-1}\Big)\vec c-\vec x=0. \tag{5.14}$$

Multiplying by $(\mathbb{1}-A)$ gives

$$\big(\mathbb{1}-A^N\big)\Big((\mathbb{1}-A)\vec x+\vec c\Big)=0, \tag{5.15}$$

and one deduces that either $\det\big(\mathbb{1}-A^N\big)\neq 0$, in which case

$$\vec x=(\mathbb{1}-A)^{-1}\vec c=\frac{1}{2-\alpha}\begin{pmatrix}\delta\\ \delta\end{pmatrix}=\begin{pmatrix}\mu\\ \mu\end{pmatrix}, \tag{5.16}$$

– a fix point of the transformation (i.e. $A\begin{pmatrix}\mu\\ \mu\end{pmatrix}+\vec c=\begin{pmatrix}\mu\\ \mu\end{pmatrix}$, making all d's and $\tilde d$'s equal to one another, i.e. $Z\equiv 0$, which we don't want) – or $2N\theta$ is a multiple of 2π, i.e.

$$\hbar=\tan\left(\frac{\pi}{N}k\right), \tag{5.17}$$

in which case (5.14) holds for *every* $\vec x$, as then $\mathbb{1}+A+\cdots+A^{N-1}=0$ and $A^N=\mathbb{1}$ (the eigenvalues of A are $e^{\pm 2i\theta}$). Expressing the constraint (5.4) in terms of

$$\begin{aligned} D&=WW^\dagger=X^2+Y^2-i[X,Y]\\ \tilde D&=W^\dagger W=X^2+Y^2+i[X,Y],\end{aligned} \tag{5.18}$$

giving

$$(D+\widetilde{D}-2\mu)^2+\frac{(D-\widetilde{D})^2}{\tan^2\theta}=4\cdot\mathbb{1}, \tag{5.19}$$

one sees that the transformation (5.10) must leave invariant the ellipse given via (5.19). Although this becomes obvious in the "circle-coordinates"

$$\begin{aligned} Z &= \frac{D-\widetilde{D}}{2\hbar}=-\frac{i}{\hbar}[X,Y] \\ \varphi &= \frac{1}{2}(D+\widetilde{D})-\mu = X^2+Y^2-\mu, \end{aligned} \tag{5.20}$$

in which the transformation (5.13) between eigenvalue-pairs is simply a rotation by 2θ,

$$\begin{pmatrix} z_{n+1} \\ \varphi_{n+1} \end{pmatrix} = \begin{pmatrix} \cos(2n\theta) & -\sin(2n\theta) \\ \sin(2n\theta) & \cos(2n\theta) \end{pmatrix} \begin{pmatrix} z_1 \\ \varphi_1 \end{pmatrix}, \tag{5.21}$$

the picture of a by 45° rotated ellipse (with halfaxes 1 and $\hbar=\tan\theta$) lying in the $(d,\widetilde{d})$-plane is extremely useful, in particular when discussing the (μ,θ)-dependence of N-dimensional representations of eqs (5.1)–(5.4) (s.b.).

Remembering, e.g. that $D=WW^\dagger$ we get

$$|W_k|^2=d_k,$$

which is only solvable if the ellipse defined by $\vec{x}_1=(d,\widetilde{d})$ entirely lies in the first quadrant. This observation leads to the following: Assume that $\theta=\pi/N$ for some $N>0$ and let $\vec{x}=(d,\widetilde{d})$ lie on the ellipse $(d+\widetilde{d}-2\mu)^2+(d-\widetilde{d})^2/\hbar^2=4$. If $\mu\le 1$ then there exists no $\vec{x}=(d,\widetilde{d})$ such that $d_n>0$ and $\widetilde{d}_n>0$ for $n=1,2,\ldots,N$. If $\mu>1$ and $\cos\theta>1/\mu$ then, for every choice of $\vec{x}=(d,\widetilde{d})$ on the ellipse, $d_n>0$ and $\widetilde{d}_n>0$ for all $n\ge 1$.
These solutions take the form

$$\begin{aligned} X &= \frac{1}{2}\begin{pmatrix} 0 & x_1 & 0 & \cdots & 0 & x_N \\ x_1 & 0 & x_2 & \cdots & 0 & 0 \\ 0 & x_2 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{N-2} & 0 & x_{N-1} \\ x_N & 0 & \cdots & 0 & x_{N-1} & 0 \end{pmatrix} \\ Y &= -\frac{i}{2}\begin{pmatrix} 0 & y_1 & 0 & \cdots & 0 & -y_N \\ -y_1 & 0 & y_2 & \cdots & 0 & 0 \\ 0 & -y_2 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -y_{N-2} & 0 & y_{N-1} \\ y_N & 0 & \cdots & 0 & -y_{N-1} & 0 \end{pmatrix} \\ Z &= \operatorname{diag}(z_1,z_2,\ldots,z_N) \\ x_l &= y_l = \sqrt{\mu+\frac{\cos\left(\frac{2\pi l}{N}+\beta\right)}{\cos(\pi/N)}} = w_l \\ z_l &= -\sin\left(\frac{2\pi l}{N}-\frac{\pi}{N}+\beta\right) \end{aligned} \tag{5.22}$$

and the ellipse, on which $(d_i,\widetilde{d}_i)$ lie, will typically look like in Figure 1.
In the region $1<\mu<1/\cos\theta$ the set of $(d,\widetilde{d})$, for which $d^{(n)},\widetilde{d}^{(n)}>0$ for all n, is a union of disjoint intervals whose lengths decreases and eventually become a set of N distinct points as $\mu\to 1$ (however,

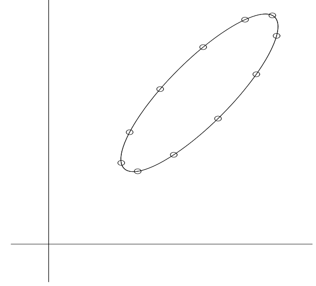

Figure 1: $\mu = 2,\ N = 11$.

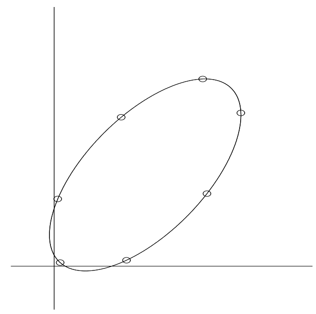

Figure 2: $\mu \approx 1.055,\ N = 7$.

making $d^{(i)} = 0$ for some i, giving not a "loop", but a "string" solution. This transition will be discussed in Section 6). For a loop solution in this region, the points $(d_i, \widetilde{d}_i)$ will precisely "miss" the negative region, like in Figure 2. For $\mu < 1$, by making a "string" Ansatz for W

$$W = \begin{pmatrix} 0 & w_1 & 0 & \cdots & 0 \\ 0 & 0 & w_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & w_{N-1} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{5.23}$$

one derives, from $D = WW^\dagger$ and $\widetilde{D} = W^\dagger W$, the condition $\widetilde{d}_1 = 0 = d_N$, which makes the choice $d_1 = 2\sin\theta\big(\mu\sin\theta + \sqrt{1-\mu^2\cos^2\theta}\big)$ necessary. Can we now, for given N, find θ such that $d_N = 0$? Assume that $-1 < \mu < 1/\cos\theta$ and let $\vec{x}_1 = \big(2\sin\theta\big(\mu\sin\theta + \sqrt{1-\mu^2\cos^2\theta}\big), 0\big)$. If θ is a solution of

$$\frac{\cos(N\theta)}{\cos\theta} = -\mu, \tag{5.24}$$

then $d_n = 0$ and $d_n, \widetilde{d}_n > 0$ for $n = 2, 3, \ldots, N-1$.

There are three values of μ for which it is particularly easy to calculate these solutions explicitly: $\mu = 1, \mu = \hbar$ and $\mu = 0$.

$\mu = 1$:

$$X = \frac{1}{2}\begin{pmatrix} 0 & x_1 & 0 & \cdots & 0 \\ x_1 & 0 & x_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & x_{N-2} & 0 & x_{N-1} \\ 0 & 0 & 0 & x_{N-1} & 0 \end{pmatrix}, \quad Y = -\frac{i}{2}\begin{pmatrix} 0 & y_1 & 0 & \cdots & 0 \\ -y_1 & 0 & y_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & -y_{N-2} & 0 & y_{N-1} \\ 0 & 0 & 0 & -y_{N-1} & 0 \end{pmatrix}$$
$$Z = \mathrm{diag}\left(z_1, z_2, \ldots, z_N\right), \quad z_l = \sin\left(\frac{2\pi l}{N+1}\right)$$
$$x_l = y_l = \sqrt{1 - \frac{\cos\left(\frac{(2l+1)\pi}{N+1}\right)}{\cos\frac{\pi}{N+1}}} \tag{5.25}$$

$\mu = \hbar = \tan\frac{\pi}{2(N-1)}$:

$$X = \frac{1}{2}\begin{pmatrix} 0 & x_1 & 0 & \cdots & 0 \\ x_1 & 0 & x_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & x_{N-2} & 0 & x_{N-1} \\ 0 & 0 & 0 & x_{N-1} & 0 \end{pmatrix}, \quad Y = -\frac{i}{2}\begin{pmatrix} 0 & y_1 & 0 & \cdots & 0 \\ -y_1 & 0 & y_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & -y_{N-2} & 0 & y_{N-1} \\ 0 & 0 & 0 & -y_{N-1} & 0 \end{pmatrix}$$
$$Z = \mathrm{diag}\left(z_1, z_2, \ldots, z_N\right), \quad z_l = \cos\left(\frac{(l-1)\pi}{N-1}\right)$$
$$x_l = y_l = \sqrt{\frac{2}{\cos\frac{\pi}{2(N-1)}}}\sqrt{\sin\left(\frac{\pi l}{2(N-1)}\right)\cos\left(\frac{\pi(l-1)}{2(N-1)}\right)} \tag{5.26}$$

$\mu = 0$:

$$X = \frac{1}{2}\begin{pmatrix} 0 & x_1 & 0 & \cdots & 0 \\ x_1 & 0 & x_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & x_{N-2} & 0 & x_{N-1} \\ 0 & 0 & 0 & x_{N-1} & 0 \end{pmatrix}, \quad Y = -\frac{i}{2}\begin{pmatrix} 0 & y_1 & 0 & \cdots & 0 \\ -y_1 & 0 & y_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & -y_{N-2} & 0 & y_{N-1} \\ 0 & 0 & 0 & -y_{N-1} & 0 \end{pmatrix}$$
$$Z = \mathrm{diag}\left(z_1, z_2, \ldots, z_N\right), \quad z_l = \cos\left(\frac{l\pi}{N} - \frac{\pi}{2N}\right)$$
$$x_l = y_l = \sqrt{\frac{1}{\cos\frac{\pi}{2N}}\sin\left(\frac{l\pi}{N}\right)} \tag{5.27}$$

In the region $-1 < \mu \leq 1$ the corresponding ellipse for a string solution will typically look like Figure 3.

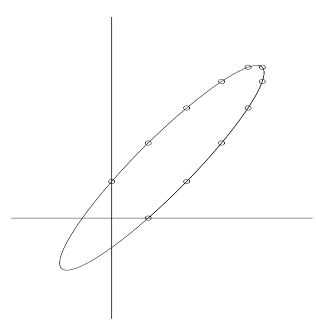

Figure 3: $\mu = 1/2$, $N = 11$ and $\theta \approx 0.189$.

Let us now derive the subcritical condition for existence of an N-dimensional string representation,

$$\cos(N\theta) + \mu\cos\theta = 0, \tag{5.28}$$

valid for all $\mu \in (-1, 1/\cos\theta]$; with no solution for $\mu = -1$ as $(X^2 + Y^2 + 1)^2 + Z^2 = 0$ cannot have any nontrivial solutions. It is useful to remember that

$$\begin{aligned} \mu = 1 \quad &\Rightarrow \quad \theta = \frac{\pi}{N+1} \\ \mu = 0 \quad &\Rightarrow \quad \theta = \frac{\pi}{2N} \\ \mu = \hbar \quad &\Rightarrow \quad \theta = \frac{\pi}{2(N-1)}. \end{aligned} \tag{5.29}$$

For $\mu \in [\tan\theta, 1]$ (other values of μ can be treated analogously):

$$\begin{aligned} d_1 &= 2\mu\sin\theta\left(\sin\theta + \sqrt{\frac{1}{\mu^2} - \cos^2\theta}\right) \quad , \quad \widetilde{d}_1 = 0 \\ d_N &= 0 \quad , \quad \widetilde{d}_N = d_1 \end{aligned} \tag{5.30}$$

gives

$$\begin{aligned} z_1 &= \frac{\cot\theta}{2} d_1 = \mu\cos\theta\left(\sin\theta + \sqrt{\frac{1}{\mu^2} - \cos^2\theta}\right) = -z_N \\ \varphi_1 &= \frac{d_1}{2} - \mu = \mu\left(\sin\theta\sqrt{\frac{1}{\mu^2} - \cos^2\theta} - \cos^2\theta\right) = \varphi_N < 0. \end{aligned} \tag{5.31}$$

Let χ be the angle from (z_1, φ_1) to (z_N, φ_N): If we, by doing $(N-1)$ rotations by 2θ, want to go from

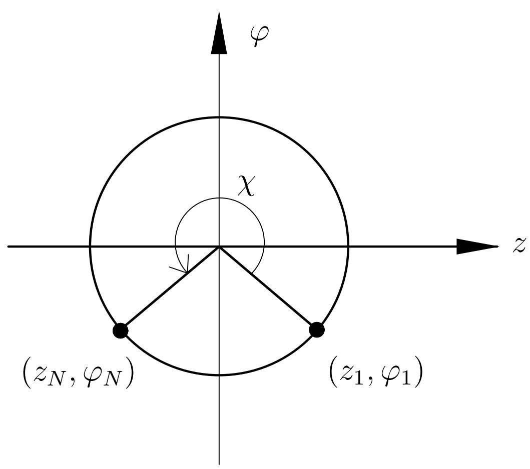

Figure 4:

(z_1, φ_1) to (z_N, φ_N), we get the condition

$$2\theta(N-1) = \pi + 2\arccos z_1 \tag{5.32}$$

since $\mu \in [\tan\theta, 1]$. Rearranging (5.32), and inserting the expression for z_1 one obtains, by taking cos of both sides

$$\sin(N-1)\theta - \mu\cos\theta\sin\theta = \cos\theta\sqrt{1 - \mu^2\cos^2\theta}. \tag{5.33}$$

Squaring the expression and solving for μ yields

$$\begin{aligned} \mu &= \tan\theta\sin(N-1)\theta \pm \sqrt{1 - \sin^2(N-1)\theta} \\ &= \mp\frac{\cos\left((N-1\pm 1)\theta\right)}{\cos\theta}. \end{aligned} \tag{5.34}$$

By knowing the sign of $\cos(N\theta)$ and $\cos\theta$, we see that one of the roots is a false root, leaving

$$\mu = -\frac{\cos(N\theta)}{\cos\theta}. \tag{5.35}$$

For given μ, out of the solutions for θ, it is only the smallest that gives the string solution; the larger θ's correspond to a total rotation of more than 2π (giving negative d_i's).

6 The singularity

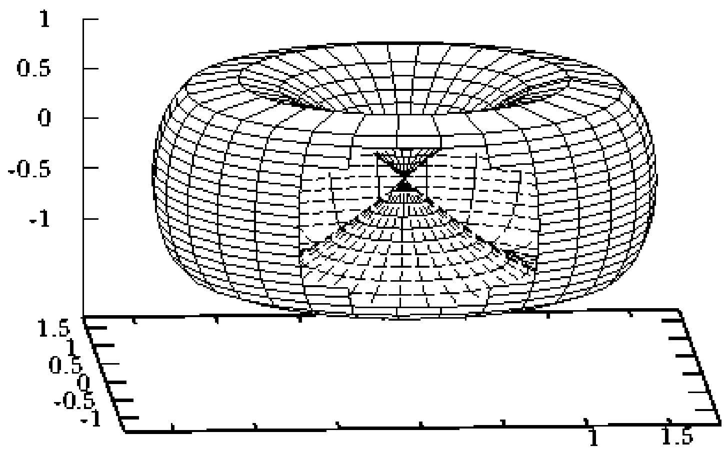

Figure 5: $(x^2 + y^2 - 1)^2 + z^2 = 1$

The equation

$$(x^2 + y^2 - \mu)^2 + z^2 = 1 \tag{6.1}$$

defines a regular surface, except for $\mu = 1$ (the singularity being at the origin, $x = y = z = 0$, see figure 5). How is this singularity reflected in the "fuzzy world", i.e. when looking at finite dimensional representation of (5.1-5.4)? Interestingly, $\Sigma_{\hbar=\tan\theta}(P = x^2 - 1)$ "does exist", for θ an integer fraction of π. Equations (5.1-5.4) do have, for $\theta = \pi/N$, (up to conjugation) a unique ("irreducible") $N-1$ dimensional solution:

$$X = \frac{1}{2}\begin{pmatrix} 0 & x_2 & & \\ x_2 & 0 & \ddots & \\ & \ddots & \ddots & x_{N-1} \\ & & x_{N-1} & 0 \end{pmatrix}, \qquad Y = -\frac{i}{2}\begin{pmatrix} 0 & y_2 & & \\ -y_2 & 0 & \ddots & \\ & \ddots & \ddots & y_{N-1} \\ & & -y_{N-1} & 0 \end{pmatrix},$$

$$Z = \begin{pmatrix} z_2 & & & 0 \\ & z_3 & 0 & \\ & 0 & \ddots & \\ 0 & & & z_{N-1} \end{pmatrix} \tag{6.2}$$

$$x_l = y_l = \sqrt{1 - \frac{\cos\left(\frac{2\pi l}{N} - \frac{\pi}{N}\right)}{\cos\left(\frac{\pi}{N}\right)}}, \quad z_l = \sin\left(\frac{2\pi(l-1)}{N}\right), \quad \text{for } l = 2, 3, \ldots, N.$$

What happens, is the following: while for large enough μ ($\mu > 1/\cos\theta > 1$), the ellipse lies entirely in the first quadrant of the $(d, \widetilde{d})$ plane ($d, \widetilde{d} > 0$), leading to an N-dimensional representation (if $N\theta = 2\pi$) for arbitrary initial conditions (i.e. an arbitrary initial point on the ellipse), this is no longer the case when μ approaches 1; when μ becomes smaller than $1/\cos\theta$, and approaches 1 from

above, the continuous range of initial conditions gradually shrinks, leaving at $\mu = 1$ precisely *one* "N-dimensional" representation, which however has the additional feature that in the limit ($\mu = 1$) the first row and column of X, Y and Z becomes identically zero. This drop of the dimensionality (by 1) for given $\theta = \pi/N$ could be viewed as reflecting the singularity, while on the other hands it cleverly ("smoothly") leads over to the subcritical ($\mu < 1$) N-dependence of θ.
The only representation that survives as $\mu \to 1$ is the one given by

$$X = \frac{1}{2}\begin{pmatrix} 0 & x_1 & 0 & x_N \\ x_1 & 0 & \ddots & 0 \\ 0 & \ddots & \ddots & x_{N-1} \\ x_N & 0 & x_{N-1} & 0 \end{pmatrix}, \qquad Y = -\frac{i}{2}\begin{pmatrix} 0 & y_1 & 0 & -y_N \\ -y_1 & 0 & \ddots & 0 \\ 0 & \ddots & \ddots & y_{N-1} \\ y_N & 0 & -y_{N-1} & 0 \end{pmatrix}, \tag{6.3}$$
$$Z = \operatorname{diag}(z_1, z_2, \dots, z_N)$$
$$x_l = y_l = \sqrt{\mu - \frac{\cos\left(\frac{2\pi l}{N} - \frac{\pi}{N}\right)}{\cos\left(\frac{\pi}{N}\right)}}, \quad z_l = \sin\left(\frac{2\pi(l-1)}{N}\right), \quad \text{for } l = 1, 2, \dots, N.$$

i.e. $d_1 = \widetilde{d_1} = \mu - 1$ (the lower tip of the ellipse) as the starting point (due to $\widetilde{d_2} = d_1 = \mu - 1 > 0$ it is qualitatively clear, that, with that initial condition one "jumps" over the small region of negative $\widetilde{d}$, as well as, "at the end" the one of negative d).
The drop in dimensionality (for $\mu = 1$) then is simply the vanishing of $x_1 = y_1$ and $x_N = y_N$. One could of course relabel the points (always start with the second point, instead of the lower tip) such that $\mu = 1$ the upper $(N-1)\times(N-1)$ block has a smooth limit (and the N^{th} row/column "disappears" as $\mu \to 1$). As noted above, the subcritical behaviour of θ as a function of μ (to have a N-dimensional representation) is more involved. As the allowed part of the ellipse shrinks to zero as μ goes from 1 to -1), θ has to accordingly decrease (for fixed dimension N of the representation); for $\mu = 0$, it is equal to $\pi/2N$.
As a reflection of the classical singularity at $\mu = 1$, the quantum (fuzzy) analogue manifests itself not only (by the sudden drop in dimension) at $\mu = 1$, but also in the neighbouring region, $1 < \mu < 1/\cos\theta$, which we shall now discuss in detail: in this range,

$$d_\pm = 2\sin\theta\left(\mu\sin\theta \pm \sqrt{1 - \mu^2\cos^2\theta}\right); \tag{6.4}$$

the d-values at which the ellipse crosses the d-axis, are both (real and) positive. The corresponding points on the z–φ circle have coordinates

$$\begin{aligned} z_\pm &= \cos\theta\left(\mu\sin\theta \pm \sqrt{1 - \mu^2\cos^2\theta}\right) > 0 \\ \varphi_\pm &= \pm\sin\theta\sqrt{1 - \mu^2\cos^2\theta} - \mu\cos^2\theta < 0. \end{aligned} \tag{6.5}$$

Let us denote the angles between the negative φ axis and the 2 points (given by (6.5)) by $\psi_\pm$ (cp. Figure 2).
At $\mu = 1/\cos\theta$: $\psi_+ = \psi_- = \theta$ and at $\mu = 1$: $\psi_+^{(1)} = 2\theta$, $\psi_-^{(1)} = 0$. To find a "closed string" solution, initial conditions with angles $\psi \in (\psi_-, \psi_+)$, are forbidden (as well as those regions obtained by rotating the interval (ψ_-, ψ_+) by $2k\theta$ ($k = 1, \dots, N-1$). To the "black" regions one has to add rotation images of the corresponding part of the ellipse that extended into the negative d-region $\psi \in \left(\widetilde{\psi}_+, \widetilde{\psi}_-\right) = (-\psi_+, -\psi_-)$.
Hence one obtains

$$B := \bigcup_{k=0}^{N-1}\left(-\psi_+ + \frac{2\pi}{N} + \frac{2\pi}{N}k, \psi_+ + \frac{2\pi}{N}k\right) \tag{6.6}$$

as the forbidden ("black") region. B continuously grows from empty (at $\mu = 1/\cos\theta$) to

$$B_1 = [0, 2\pi) \setminus \left\{0, \frac{2\pi}{N}, \frac{4\pi}{N}, \dots, 2\pi\frac{N-1}{N}\right\} \tag{6.7}$$

(at $\mu = 1$, where $\varphi_+ = 2\theta = 2\pi/N$, due to $a(\varphi = 0) = 0 = a(\varphi = 0)$ at $\mu = 1$ the "closed string" solution disappears, the corresponding dimensionality drops by 1, and the "truely N-dimensional" open-string representation then corrponds to $\theta = \pi/(N-1)$).
Note that while in the "critical region" ($1 \leq \mu \leq 1/\cos\theta$) both –closed and open– string N-dimensional solutions exist, they never coexist for the same value of θ (resp.$\hbar$). N-dimensional closed-string-solutions naturally require $\theta = 2\pi/N$, while N-dimension open-string-solutions are subject to the quantisation condition

$$\cos(N\theta) + \mu \cdot \cos\theta = 0 \tag{6.8}$$

(the derivation is identical to the one for the subcritical region(s), $\mu < 1$), which gives $\theta = \pi/N$ for $\mu = 1/\cos\theta$, and $\theta = \pi/(N-1)$ for $\mu = 1$ (and no integer lying between N and $N-1$).

Acknowledgement

We would like to thank the Royal Institute of Technology, the Albert Einstein Institute, the Japan Society for the Promotion of Science, the Sonderforschungsbereich "Raum-Zeit-Materie", and the Marie Curie Research Training Network ENIGMA for financial support and hospitality.

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